Application for a Computing Time Project on the RWTH Compute Cluster

Project extension proposal for project "pmGenerator"

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An exhaustive generator to find shortest known condensed detachment proofs, focussing on completeness proofs in Hilbert systems over minimal single axioms for propositional calculus

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Abstract

Utilization of a proof generator with shared memory parallelization making heavy use of Intel's oneTBB library, and distributed memory parallelization via MPI for a computing-intensive filtering method.

As described in the original proposal[1], the tool *pmGenerator* can generate exhaustive proof collections in concise formal representations. Since version 1.2, which I officially released on March 3, 2024, after four months of testing, it allows user-defined axioms to customize systems based on rules `D` for condensed detachment and `N` for necessitation. The latter can be used to define systems of modal logic but is disabled by default. Version 1.2 introduces plenty of features meant to assist in generating proofs that are not guaranteed to be minimal but with a high emphasis on shortness. This proved to be essential in finding derivations even of short theorems in more complex propositional proof systems, which is taken to an extreme when exploring 1-bases, i.e. complete systems with only a single theorem as an axiom schema.

Following a paper from June 2021 by M. Walsh and B. Fitelson[2], which as of March 2024 is still listed as "under review", there are only seven minimal single axioms for propositional logic in terms of $\{\to, \neg\}$ under modus ponens, which is encompassed by consensed detachment. These are Meredith's popular single axiom `CCCCCpqCNrNsrtCCtpCsp` and Walsh's six axioms `CCpCCNpqrCsCCNtCrtCpt`, `CpCCqCprCCNrCCNstqCsr`, `CpCCNqCCNrsCptCCtqCrq`, `CpCCNqCCNrsCtqCCrtCrq`, `CCpqCCCrCstCqCNsNpCps` and `CCCpqCCCNrNsrtCCtpCsp`. Walsh's only remaining candidate can be ruled out by generating all of its theorems via `pmGenerator -c -n -s CCCpqCCrNsCtNtCCtpCrp -g -1` – which turn out to be only finitely many schemas via a calculation that takes a few milliseconds. I could reproduce completeness results with *pmGenerator* for the other axioms excluding Walsh's second axiom, for which I succeeded only in generating a non-constructive proof using the Vampire theorem prover, thus to find such a proof within its Hilbert system remains an open challenge. So far, I did not attempt to reproduce the claim that those seven axioms are the *only* ones of their kind, but I might in the future.

While all 1-bases take great advantage from compact condensed detachment proof notation due to very long intermediate formulas commonly occurring in their proofs, the system of Walsh's second

[1] `https://xamidi.github.io/pmGenerator/pdf/rwth1392_abstract.pdf`

[2] Preprint: `http://fitelson.org/walsh.pdf`. (As of March 2024, it still contains several mistakes and refers to an inaccessible code base, of which I informed Prof. Fitelson in September 2023 as part of an email conversation.)

axiom is an excellent example of a proof system that requires immense effort and very long formulas to arrive at short conclusions. Its shortest non-trivial proof `D11` translates to

$$
\begin{array}{|ll}
\hline
1.\ \varphi \to ((\chi \to (\varphi \to \eta)) \to ((\neg\eta \to ((\neg\tau \to \theta) \to \chi)) \to (\tau \to \eta))) & (A1) \\
2.\ (\varphi \to ((\chi \to (\varphi \to \eta)) \to ((\neg\eta \to ((\neg\tau \to \theta) \to \chi)) \to (\tau \to \eta)))) \to & \\
\quad ((\psi \to ((\varphi \to ((\chi \to (\varphi \to \eta)) \to ((\neg\eta \to ((\neg\tau \to \theta) \to \chi)) \to (\tau \to \eta)))) \to \zeta)) \to & \\
\quad\ ((\neg\zeta \to ((\neg\xi \to \sigma) \to \psi)) \to (\xi \to \zeta))) & (A1) \\
3.\ (\psi \to ((\varphi \to ((\chi \to (\varphi \to \eta)) \to ((\neg\eta \to ((\neg\tau \to \theta) \to \chi)) \to (\tau \to \eta)))) \to \zeta)) \to & \\
\quad ((\neg\zeta \to ((\neg\xi \to \sigma) \to \psi)) \to (\xi \to \zeta)) & (MP) : 1, 2
\end{array}
$$

and its shortest proof of *any* theorem smaller than its 21-symbol axiom is `DDD11DDD111DDDDD111111DDDDD111111` (33 steps), which proves the 17-symbol theorem `CpCqCCNpCCNrstCrp`. This merely indicates the magnitude of that system's complexity and how condensed detachment leads to significant savings in data to be processed, especially since formulas tend to blow up in size for increasing lengths of proofs.

An automated approach using well-optimized tools is required to explore these kinds of proof systems and to tackle questions regarding their complexity. For example, exploring systems in propositional logic might in the long run lead to useful results towards solving the NP versus coNP problem in the field of proof complexity.

Keywords: Logic, Proof theory, Hilbert systems, Condensed detachment

Achieved Results

Apart from supporting the development and testing of the free and open-source software project *pmGenerator*[3], this computing time project generated a lot of knowledge in the past year, including but not limited to:

- Five shorter proofs in the “Shortest known proofs of the propositional calculus theorems from Principia Mathematica”[4] collection of Metamath.
- Exhausted proof length increments in databases for nine proof systems:
  - i. Frege’s calculus simplified by Łukasiewicz (480.99 GB): $35 \mapsto 39$
  - ii. Meredith’s 1-basis (434.50 GB): $65 \mapsto 83$
  - iii. Walsh’s 1-bases (499.54, 383.44, 412.84, 815.79, 404.05, 470.47 GB): $135 \mapsto 161$, $33 \mapsto 43$, $59 \mapsto 73$, $147 \mapsto 169$, $47 \mapsto 55$, $77 \mapsto 95$
  - iv. S5 (standard modal extension of i.) (218.25 GB): $22 \mapsto 30$
- Lists of smallest 1000 theorems with known minimal proofs for all of the above systems. These can be found linked in the project’s readme[5].
- Abstract proof summaries of shortest known completeness proofs for Meredith’s axiom and Walsh’s axioms, except Walsh’s second axiom, whose file is missing a proof for `CCpqCCqrCpr` towards completeness.
  - This led to the launch of a corresponding proof minimization challenge[6], in which everyone is welcome to participate.

Computations to build exhaustive proof databases as part of this project provided valuable insights in terms of memory and CPU usage, data growth rates and information density. Great variations in seemingly similar systems may appear particularly striking. This information can be found in the project’s readme, along with log files that resulted from computations, and download links to the compressed databases.

[3]https://github.com/xamidi/pmGenerator
[4]https://us.metamath.org/mmsolitaire/pmproofs.txt
[5]https://xamidi.github.io/pmGenerator/README.html
[6]https://github.com/xamidi/pmGenerator/discussions/2