Application for a Computing Time Project on the RWTH Compute Cluster

Project extension proposal for project “pmGenerator”

Period: 01.04.2025 – 31.03.2026

An exhaustive generator to find shortest known condensed detachment proofs, focussing on correlative proofs between Hilbert systems over minimal single axioms for propositional calculus

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Abstract

Utilization of a proof generator with shared memory parallelization making heavy use of Intel's oneTBB library, and distributed memory parallelization via MPI for a computing-intensive filtering method.

As described in the original proposal[1], the tool *pmGenerator* can generate exhaustive proof collections in concise formal representations. Since version 1.2, released on March 3, 2024, it allows user-defined axioms to customize systems based on rules `D` for condensed detachment and `N` for necessitation. The latter can be used to define systems of modal logic and is disabled by default. Plenty of features to assist with generating shorter proofs from longer known proofs were added up to the current version towards the 1.2.2 release. This includes fully automated proof compression algorithms that led to tremendous success in proof minimization challenges – as presented under "Achieved Results" – building upon achievements mentioned in the previous extension proposal.[2] Extensive automation proved to be very useful to explore complex proof systems, especially the previously mentioned minimal 1-bases for propositional logic in terms of $\{\rightarrow, \neg\}$ under modus ponens, which is encompassed by condensed detachment. These 1-bases are Meredith's single axiom `CCCCCpqCNrNsrtCCtpCsp` and Walsh's six axioms `CCpCCNpqrCsCCNtCrtCpt`, `CpCCqCprCCNrCCNstqCsr`, `CpCCNqCCNrsCptCCtqCrq`, `CpCCNqCCNrsCtqCCrtCrq`, `CCpqCCCrCstCqCNsNpCps` and `CCCpqCCCNrNsrtCCtpCsp`. I could reproduce completeness results with *pmGenerator* for all of these systems, which includes finding the first constructive completeness proof for Walsh's second axiom on July 12, 2024. This solved an open challenge problem that was established by Walsh and Fitelson on June 26, 2021.[3] Furthermore, I found surprisingly *short* constructive proofs from these 1-bases. For example, a 227-step derivation from Walsh's second axiom resulting in Łukasiewicz's axiom `CCNpNqCqp`, for which previously no constructive proof could be found. The latter seems to be due to two conjectures:

1. *Walsh's second axiom requires some long intermediate formulas to arrive at certain short conclusions in feasible amounts of steps.* But current automated theorem provers have no way of probably finding long formulas that are useful intermediate conclusions.

[1]`https://xamidi.github.io/pmGenerator/pdf/rwth1392_abstract.pdf`

[2]`https://xamidi.github.io/pmGenerator/pdf/rwth1392_extension_2024.pdf`

[3]Preprint: `http://fitelson.org/walsh.pdf`. (As of March 2025, it still contains several mistakes and refers to an inaccessible code base, of which I informed Prof. Fitelson in September 2023 as part of an email conversation.)

2. *Minimal proofs of the desired theorems are significantly longer than their corresponding exhaustion barrier of closely above 43.* Each proof system has an *exhaustion barrier*, which is the maximum number of proof steps for which an exhaustive search is possible due to current technological limitations. Increasing an exhaustion barrier requires exponentially increased resources under deterministic computing models.

For instance, the shortest known proofs from Walsh's second axiom towards the widespread complete systems `CCpqCCqrCpr,CCNppp,CpCNpq` (703, 141, 57 steps) and `CpCqp,CCpCqrCCpqCpr,CCNpNqCqp` (53, 1111, 227 steps) contain 90 and 111 different conclusions of approximate sizes 71.84 and 70.36 on average, respectively. The biggest formula has 322 symbols in both cases, which is still far below average: The most general theorems with minimal proofs of length 37 are already sized approx. 1173.67 symbols on average, and this grows for longer proofs.

So far, I did not attempt to reproduce the claim that those seven axioms are the *only* ones of their kind, but I might in the future. Thanks to the swift advancement of the minimal 1-bases completeness proof minimization challenge, another challenge soon to be tackled is the derivation of single axioms from each other and using proof compression techniques in order to estimate their relative complexities.

Keywords: Logic, Proof theory, Hilbert systems, Condensed detachment

Achieved Results

Apart from supporting the development and testing of the free and open-source software project *pmGenerator*[4], this computing time project generated a lot of knowledge in the past year, including but not limited to:

- Sixty-two proofs reduced by 2144 steps in the "Shortest known proofs of the propositional calculus theorems from Principia Mathematica"[5] collection of Metamath. This elevates my overall contribution to this database to a total of sixty-five proofs reduced by 3034 steps.

- Twenty-four proofs reduced by 5021154 steps in the "Minimal 1-bases for C-N propositional calculus"[6] proof minimization challenge.

  - Additionally introduced two proofs of 2602085 and 1228561 steps for ($A2$): `CCpCqrCCpqCpr` and ($L1$): `CCpqCCqrCpr` from Walsh's second axiom, completing the challenge of finding a constructive completeness proof, as mentioned in the abstract. This built upon a 1877-step proof for ($A3$): `CCNpNqCqp` found by Steven Nguyen.
  - Furthermore, five proofs were reduced by 1996792 steps by other people as part of the challenge.

- Conceptualization of behavioral graphs, meant to provide fingerprints of proof systems by illustrating their amounts of minimally proven theorems of certain sizes relative to proof lengths that approach their exhaustion barrier for filtered minimal proof collections.
  I created such graphs for nine systems of interest. These are displayed in the project's readme[7] that also provides their raw plot data, which was obtained with the `pmGenerator --plot` command.

- Pioneered and implemented two condensed detachment proof compression algorithms, one to find a minimal proof relative to a set of intermediate conclusions when proofs for those are given, and another algorithm building on the former one to additionally build and utilize exhaustive proofs from all given subproofs up to an abstract length specified by a user-defined parameter.

[4]https://github.com/xamidi/pmGenerator
[5]https://us.metamath.org/mmsolitaire/pmproofs.txt
[6]https://github.com/xamidi/pmGenerator/discussions/2
[7]https://xamidi.github.io/pmGenerator/README.html